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THE MECHANICAL FUNDAMENTALS AND APPLICATIONAL MODES OF INERTIAL NAVIGATION

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THE MECHANICAL FUNDAMENTALS AND APPLICATIONAL MODES OF INERTIAL NAVIGATION

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ABSTRACT /1*

The author has undertaken in a very general form an exposition of the principles of rational mechanics which constitute the basis of inertial navigation. He has shown how closely the accuracy characteristic of this form of navigation is linked with the structure of the gravitational field in which the vehicle moves. Applications to ballistic missile guidance, aircraft and ship detection, and space mission control are discussed briefly in the light of general principles, with a detailed technical description avoided.

1. Introduction

We are all familiar with the episode, occasionally found in detective fiction, in which the kidnap victim, transported blindfolded in an automobile, succeeds in reconstructing and retracing the route travelled on the basis of the muscular impressions experienced during turns and directional changes, as well as through the actions of the driver on the brake pedal and accelerator. This process, a not altogether fanciful one, provided the victim is well acquainted with the region in question, constitutes a /2
rudimentary effort at inertial guidance.

In a more elaborate form, the precise measurement of accelerations through an internal procedure and their double integration with a view toward the derivation of the velocity and positional variables has long appeared to be an elegant, if somewhat utopian, method of fixing the position of a moving object. Inertial navigation in its most radical embodiment, that is when it borrows nothing from other navigational techniques, is essentially based on experimental manipulations effected on board the vehicle, and requires no links of any kind with the outside world. It is, therefore, in effect "endonavigation", to use the term which was first applied to it -

*Numbers given in the margin indicate the pagination in the original foreign text.

we believe - by Professor Roccard and which perhaps defines it more nicely than its customary appellation.

The autonomy conferred on the vehicle by inertial navigation is quite obviously a particularly attractive feature in the case of military applications. It is precisely these applications, in fact, which have led to the major technological effort, the result of which, for this technique, can be seen in the spectacular breakthroughs which have been achieved in limited but important areas.

It has occurred to us that it might be of interest to present to the Association, without undertaking a detailed description of instrumentation - something both beyond the desirable scope of a paper of this type and outside the competence of its author - , some reflections of a general nature with respect to the mechanical fundamentals of inertial navigation, the principal difficulties encountered in its implementation, the forms which it may assume according to its varying applications, and finally its prospects for further development.

2. Fundamental Principles

2.1.

Given a vehicle (Fig. 1), to which we assign the triangulation reference point O_1, x, y, z, we shall understand by navigation the continuous and instantaneous determination of the coordinates of one point of the vehicle, O_1 for example, with respect to an external marker Oxyz, which we shall assume to be Galilean. The anticipated precision of the operation is such, moreover, that in general no interest attaches to any discussion of the precise selection of point O_1. Navigation in the context, therefore, is limited to a real-time location of the moving object. It is, of course, quite true that the idea of navigation, as generally understood, brings to mind a more extensive operation, involving the actions which the navigator, apprised of his position, exerts upon the movement of his vessel with an eye toward rendering that movement consistent with his desire or with the requirements of the mission which has been entrusted to him. Despite the linguistic (or semantic) ambiguity inherent in this usage, we prefer to reserve the term "guidance" for this all-encompassing operation, while employing the word "navigation" in its narrowest sense, i.e., that of the determination of the fix, to borrow an expression from naval parlance. /3

Suppose that within the vehicle we place a small test mass m, free of all connection (in the sense of rational mechanics) with the vehicle, but subject to a single force F, which is perfectly controllable and measurable by an on-board operator. Suppose further that this operator (or a mechanical mechanism acting in his stead) possesses reflexes fast enough to keep mass m continuously within the vehicle through proper manipulation of force F. If care is taken to note the value of $\vec{F}$ at each instant, the acceleration $\vec{\gamma}$ of the mass m will be known through $\vec{\gamma} = \frac{\vec{F}}{m}$, and if the initial position and velocity of this mass are known, it will be possible to calculate, at any instant through double integration, the absolute position of mass m, thus of a point interior to the vehicle, thus of the vehicle itself. This computation is expressed by the equation

$$\frac{d^2\vec{r}}{dt^2} = \vec{\gamma}(t) \qquad (1)$$

where $\vec{r}$ is the vector linking the origin to point P occupied by m and the derivation is carried out with respect to the absolute reference.

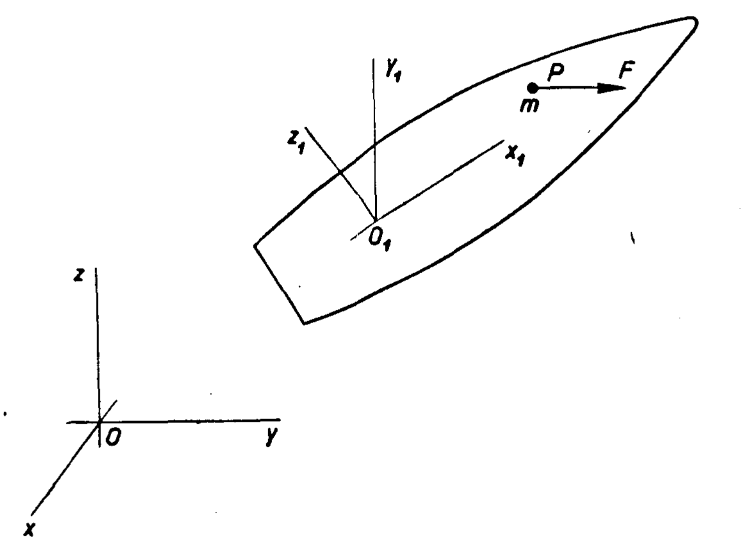


Fig. 1

2.2 /4

Examining the matter a bit more closely, it is evident that the success of the preceeding operation raises three fundamental questions.

1) It is first of all essential that the measurements and computations be carried out with a degree of accuracy sufficient to ensure that the accuracy of the result will be acceptable, despite the clearly cumulative nature of the errors.

2) In the second place, it should be noted that the acceleration $\vec{\gamma}$ can be measured only by its components in the trihedral based on the vehicle, while the integration must deal with the components in the fixed trihedral. It is thus important to know the orientation of the moving trihedral with respect to the fixed. Positional or linear inertial navigation presupposes the solution of a roblem of orientational or angular navigation. If the principle of endonavigation is to be safeguarded and maintained throughout, this operation must itself resort to inertial procedures. On the other hand, if we are willing to accept a departure from this principle with respect to this

particular point, then a very general procedure which suggests itself for the determination of the orientation of the moving trihedral might take the form of star sightings, that is the measurement of the directional parameters of two stars within the moving trihedral.

3) Thirdly, the procedure presupposes an ability to measure, if not to control, the forces of every kind which are brought to bear on mass m. If among these forces there is an uncontrollable but known force, it can always be compensated by the action of the controllable forces. On the other hand, if they include an unknown force, the method is not workable.

2.3

Disregarding for the time being the question of precision and operating on the assumption that the angular navigation problem has been resolved through the application of an arbitrary procedure, let us consider this last point with greater attention. Among the forces naturally brought to bear on mass m, certain of them originate at points which are part of the system; in principle, a knowledge of these forces presents no difficulties. Others are remote actions emanating from points outside the system. Among this latter category, the intervention of magnetic or electrical forces can always be avoided through the selection of a test mass containing no magnetic or electrical charge; gravitational forces, however, cannot be eliminated.

Let $\vec{g}$ be the gravity field at point P which is the site of mass m*. With F representing the force controlled by the navigator, the equation of motion for P is written: /5

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F} + m\vec{g}$$

or, continuing to set $\frac{\vec{F}}{m} = \vec{\gamma}$,

$$\frac{d^2\vec{r}}{dt^2} = \vec{\gamma} + \vec{g} \qquad (2)$$

$\vec{\gamma}$, the ratio of the "sensible" force F to the test mass m, has the dimensions of an acceleration. We shall refer to it as "sensible acceleration."

Let us consider the consequences of the above from the point of view of navigation.

- If we know nothing concerning the gravitational field $\vec{g}$, it is impossible to proceed from $\vec{\gamma}$ to the actual acceleration $\frac{d^2\vec{r}}{dt_2}$. Inertial techniques are

*A more precise expression would be "the field of Newtonian attraction due to masses in the universe other than those which are part of the vehicle." Practically speaking, moreover, the vehicle mass attraction is negligible.

therefore of no value to a navigator moving in an unknown gravitational field.

- Conversely, if the gravitational field is known at every point, that is if we accept $\vec{g}(\vec{r})$ as a given function, Eq. (2) assumes the form:

$$\frac{d^2\vec{r}}{dt^2} - \vec{g}(\vec{r}) = \vec{\gamma}(t). \qquad (3)$$

Vector $\vec{\gamma}$ is a priori unknown, but measurable. It thus takes the form of a given function of time.

Vector $\vec{g}$ is not measurable, but known as a priori as a function of space.

The problem of inertial navigation is therefore solvable. It reduces to the solution of a second-order differential equation (three scalar equations) in which the measured acceleration $\vec{\gamma}$ serves as the second term.

2.4

The foregoing conclusions can be presented in somewhat more scientific language by turning to the principles of the mechanics of relative motion, a review of which, in any event, seems not inappropriate.

We are aware that if we distinguish at point P the mass m itself and the material position of the vehicle with which it coincides at the instant considered, the theorem for the composition of accelerations, derived from pure kinematics, /6 is expressed by the equality:

$$\vec{\gamma}_a = \vec{\gamma}_r + \vec{\gamma}_e + 2\vec{\omega}_e \wedge \vec{V}_r$$

where $\vec{\gamma}_a$ is the absolute acceleration and γ_r is the relative acceleration of mass m,

$\vec{\gamma}_e$ is the drive acceleration (impulsive acceleration), that is the absolute acceleration of the vehicle coinciding with m,

$\vec{\omega}_e$ is the angular drive velocity (angular impulsive velocity), that is the angular velocity of the moving trihedral,

$\vec{V}_r$ is the relative velocity of mass m.

The fundamental dynamics relation applied to point m

$$\vec{F} + m\vec{g} = m\vec{\gamma}_a$$

is written as follows:

$$\vec{F} + m[\vec{g} - \vec{\gamma}_e - 2\vec{\omega} \wedge \vec{V}_r] = m\vec{\gamma}_r$$

and may be interpreted through the following statement:

"The fundamental dynamic relation $\vec{F} = m\vec{\gamma}$ is applicable to movement determined with respect to any trihedral, on the condition that for the gravitational field due to Newtonian attraction alone there is substituted an apparent gravitational field defined by:

$$\vec{g}' = \vec{g} - \vec{\gamma}_e - 2\vec{\omega} \wedge \vec{V}_r.$$

This field has the particular feature of encompassing a term which is a function of the relative velocity, since the driving (impulsive) movement is not purely translational.

If in the procedure described above the conditions were to be made more severe by requiring that mass m remain in a state of relative rest within the vehicle (which would necessitate greater vigilance on the part of the operator or a negligible degree of error), the acceleration of the mass and that of the coincident point could be regarded as identical. The instrument which until this moment was measuring only its own sensible acceleration is now measuring the acceleration of a specific point within the vehicle. This instrument is called an accelerometer. It is important, however, not to lose sight of the fact that, despite its misleading name, the accelerometer does not actually measure the absolute acceleration, but rather the sensible acceleration $\vec{\gamma}_e - \vec{g}$ or, by simply changing the sign of the scalar, the apparent gravity $\vec{g} - \vec{\gamma}_e$. The impossibility of separating $\vec{g}$ and $\vec{\gamma}_e$ by means of an internal operation is an unavoidable consequence of the identity of the gravitational and inertial masses.

Let us further note that if the accelerometer is positioned at the inertial center of the vehicle, its readings are subject to another interesting interpretation. If $\vec{\mathcal{F}}$ is the resultant of the sensible external forces acting on the vehicle of mass M, according to the theorem for the movement of the center of inertia we have: /7

$$\vec{\mathcal{F}} + M\vec{g} = M\vec{\gamma}_e$$

whence

$$\vec{\gamma}_e - \vec{g} = \frac{\vec{\mathcal{F}}}{M}.$$

Thus, the accelerometer measures that part of the vehicle's acceleration which is due to non-gravitational forces or, on another scale, the resultant itself of these forces.

2.5

Let us return to the basic equation of inertial navigation:

$$\frac{d^2\vec{r}}{dt^2} - \vec{g}(\vec{r}) = \vec{\gamma}(t)$$

Mere inspection of this formula suggests the thought that inertial navigation will assume quite different aspects according to the order of magnitude of the two terms found in the first member. From this point of view, we are confronted with two extreme cases. The first case is that of $\vec{g} = 0$. In the absence of gravitational force, Eq. (3) is resolved to Eq. (1):

$$\frac{d^2\vec{r}}{dt^2} = \vec{\gamma}(t) \qquad (4)$$

and the computation is reduced to a double integration.

The second extreme case obtains when the velocity $\frac{d\vec{r}}{dt}$ is constant. This situation subsumes as an even more particular case that in which the vehicle is motionless. Eq. (3) now reduces to:

$$\vec{g}(\vec{r}) = -\vec{\gamma}(t).$$

Thus, without integration and by means of an ordinary equation the method furnishes us with the position. In the final analysis, the technique consists in using the accelerometer as a gravimeter, with the determination made on the basis of the gravitational components. This procedure we shall call gravimetric navigation.

It is of interest to note that gravimetric navigation is not completely unworkable even when function $\vec{g}(\vec{r})$ is known only imperfectly or not at all, provided one accepts a less stringent definition of navigation than that which we have adopted. Should the navigator find himself not in a homogeneous space, but in an environment, where every element of which is recognizable by virtue /8 of certain permanent and unique properties, he might limit his objective solely to the identification of his position, determined in an arbitrary system of variables, even without any known relation to metric coordinates, similar to that of an address in a city. The position of a point can then be determined by the value of g at that point, even if the relation $\vec{g}(\vec{r})$ is unknown. Stated differently, the knowledge of $\vec{g}$ will enable the navigator to situate himself on a map whose geometric conformity with reality is more or less approximate, but on which have been plotted precise isovalue curves for the components of $\vec{g}$.

A strictly rigorous application of gravimetric navigation is possible if the navigator can be assured, at least at certain moments, of his immobility. There are a number of cases in which such assurance can be acquired with notable ease, although obviously only at the expense of yet another violation of the principles of endonavigation. An explorer travelling over solid ground will find it quite natural to come to a halt in order to take his position. The position with respect to the stars of the local vertical (as determined by a pendulum or by the normal to the free surface of a liquid in a state of rest), completely in keeping with the system outlined above, provides the traveller with a knowledge of his location, as long as the points of the earth are precisely identified by the direction of their vertical.

Gravimetric navigation has been effectively employed for a very long time. Through its use it has been possible to assign to every point on the earth two parameters, called latitude and longitude, which in effect characterize the direction of the force of gravity at that point.* The attribution to these points of metric coordinates, with respect to a frame of reference for our planet, which is the proper subject of geodesy, constitutes a separate problem, solved only some time later and in less precise a manner.

If the traveller is unable to come to a state of complete rest (as in the case of the navigator at sea), gravimetric navigation can still be applied in an approximate fashion. It frequently happens, in particular, that the magnitude of instantaneous acceleration $\frac{d^2\vec{r}}{dt^2}$ is too considerable to be disregarded with respect to $\vec{g}$, but that its mean value, within a well chosen time interval, is on the contrary negligible. Eq. (3) can then be reduced to Eq. (4), provided that mean values are taken for $\vec{g}$ and $\vec{\gamma}$ Gravimetric navigation can then be reintroduced by employing an accelerometer of very long-period which behaves like a lowpass filter for $\vec{\gamma}(t)$. In this manner the navigation of a ship can be assured by defining the vertical by means of a long-period pendulum, such as a Fleuriais gyroscope, and by maintaining as constant a heading and speed as possible, so as to minimize the mean value of acceleration, with the course and speed variations caused by the agitation of the sea filtered out through the action of the pendulum. /9

In all instances in which the use of these expedients provides insufficient accuracy - particularly if the mean value of $\frac{d^2\vec{r}}{dt^2}$ is not negligible - there will be no alternative but to measure $\vec{\gamma}$ as precisely as possible and to integrate Eq. (4). One thus arrives at the final form of inertial navigation. It will be observed, however, that there exists between inertial and gravimetric navigation (along with its traditional refinements) no break or discontinuity, but that in fact the former is actually a particular case of the latter.

3. Precision of Inertial Navigation

3.1

Let us turn back now to Eq. (4) and consider what degree of precision or accuracy we may expect with respect to vector $\vec{r}$ after its integration up to time t.

*The fact that the earth constitutes a non-Galilean reference marker introduces certain complications to be discussed below.

Let $\vec{r}$ be the vector which defines the true position, and $\vec{r} + \vec{\delta r}$ the vector which defines the estimated position.

Vector $\vec{r}$ is defined by Eq. (4):

$$\frac{\delta \vec{r}}{dt^2} - \vec{g}(\vec{r}) = \vec{\gamma}(t).$$

Vector $\vec{r} + \vec{\delta r}$ is determined by the computations of the navigator who measures $\vec{\gamma}$ with an error of $\vec{\delta\gamma}$ and errs with respect to $\vec{g}$ in two ways: on the one hand, he takes g at point $\vec{r} + \vec{\delta r}$ and not at point $\vec{r}$; on the other hand, he may commit a systematic error $\delta\vec{g}$ with respect to g because of an imprecise knowledge of the gravitational field. Thus we have:

$$\frac{d^2}{dt^2}(\vec{r} + \delta\vec{r}) - \vec{g}(\vec{r} + \delta\vec{r}) = \vec{\gamma}(t) + \delta\vec{\gamma})(t).$$

Whence, by subtraction and assuming the errors are small:

$$\frac{d^2}{dt^2}\delta\vec{r} - \overline{\overline{\mathbf{grad}\,g}}\,\delta\vec{r} = \delta\vec{\gamma} + \delta\vec{g} \tag{5}$$

It will be seen that the characteristic error with respect to $\vec{g}$ is added to the error with respect to $\vec{\gamma}$. Since the chances are that $\vec{g}$ will be known with far more precision than $\vec{\gamma}$ will be measured, the effect of this is rather slight.

The preceeding equation, which is a linear differential equation of time function coefficients, defines the error $r\vec{\delta}$. Its integral will obviously bring into play the initial values of position and velocity $\delta\vec{r}_0$ and $\left(\frac{d\delta\vec{r}}{dt}\right)_0$. /10

3.2

Let us limit our examination to a few simple particular cases.

Let us suppose, first of all, that the navigation is unidimensional along an axis Ox. Vector $\vec{r}$ is reduced to a component x, $\vec{g}$, and $\vec{\gamma}$ with their projections g and γ on Ox. Eq. (4) is now written:

$$\frac{d^2x}{dt^2} - g(x) = \gamma(t)$$

and the error equation:

$$\frac{d^2}{dt^2}\delta x - \frac{dg}{dx}\delta x = \delta\gamma + \delta g$$

a) <u>If the gravitational field $\vec{g}$ is uniform.</u> We have only:

$$\frac{d^2\delta x}{dt^2} = \delta(\gamma + \bar{g})$$

The error is derived by double-squaring the error for $(\gamma + g)$. If, for example, $\delta(\gamma + g) = \delta\gamma_0$ = Cont, we have:

$$\delta x = \delta x_0 + \delta \dot{x}_0 \cdot t + \frac{1}{2} \delta \gamma_0 \cdot t^2.$$

The initial position error is preserved completely, the initial velocity error gives a term at t, the error for γ gives a term at t^2.

b) <u>If the gravitational field is linear at x,</u> $\frac{dg}{dx}$ is a constant. Let us assume it to be <u>positive</u>, and set:

$$\frac{dg}{dx} = k^2.$$

The equation without second number (5) yields a general integral of the form:

$$\delta x = Ae^{kt} + Be^{-kt}$$

This means that even when there is no error with respect to γ, the position error will contain a term which increases exponentially in time. By introducing the initial values of the error δx_0 and $\delta \dot{x}_0$, we can write: /11

$$\delta x = \delta x_0 \operatorname{ch} kt + \frac{\delta \dot{x}_0}{k} \operatorname{sh} kt.$$

When an error $\delta(\gamma + g)$ is added, the equation containing the second number must be integrated.

If $\delta(\gamma + g) = \delta\gamma_0$ = Cont, we easily find:

$$\delta x = \delta x_0 \operatorname{ch} kt + \frac{\delta \dot{x}_0}{k} \operatorname{sh} kt + (\operatorname{ch} kt - 1) \frac{\delta \gamma_0}{k^2}.$$

Even in the absence of any initial error, the measurement error $\delta\gamma_0$ results in an exponentially increasing position error. In this case, just as in the one preceeding, inertial navigation may be described as <u>unstable</u>.

c) If $\frac{dg}{dx}$ is constant but negative, we set: $\frac{dg}{dx} = -k^2$.

In the absence of error with respect to γ, the position error now evolves according to the law:

$$\delta x = \delta x_0 \cos kt + \frac{\delta \dot{x}_0}{k} \sin kt$$

and for

$$\delta(\gamma + g) = \delta\gamma_0$$

$$\delta x = \delta x_0 \cos kt + \frac{\delta \dot{x}_0}{k} \sin kt + (1 - \cos kt)\frac{\delta\gamma_0}{k^2}.$$

The error δx is now oscillatory and limited in amplitude. Inertial navigation can be described as stable.

Comment I. Even in this latter case, one must be careful not to state that the limited character of $\delta\gamma$, whatever may be the function $\delta\gamma(t)$, assures the same character for error δx. The equation without the second member, in fact, contains no damping term, so that a sinusoidal error $\delta\gamma_0$ and of pulsation k would result, through resonance, in an indefinite increase in δx. However, if we assume a probabilistic viewpoint and envisage an error having a continuous frequency spectrum, this eventuality may be disregarded.

Comment II. Let us count the abscissas on Ox, beginning at the point at which g is cancelled. It will be seen at once that the error δx is equal to the abscissa of a heavy material point having at the instant t = 0 the position and initial velocity δx_0 and $\delta \dot{x}_0$ and subject, in addition to gravity, to acceleration $\delta\gamma$.

3.3 /12

By now envisaging a form of navigation of three degrees of freedom in a spatial region sufficiently limited to permit the consideration of $\overline{\overline{\text{grad}}}$ g as constant, we can always employ a reference trihedral such that this tensor contains only diagonal terms. The errors on the three axes will develop independently, following the preceeding forms, while the stability factors will follow the three directions bound to the respective values of: $\frac{\delta g_x}{\delta x}$, $\frac{\delta g_y}{\delta y}$, $\frac{\delta g_z}{\delta z}$. However, the general properties of the Newtonian field, whose divergence is zero, implies that one at least of these quantities is positive.

There always exists, therefore, in free space at least one direction in which inertial navigation is unstable.

For example, if we place ourselves in the vicinity of the Earth, inertial navigation is stable horizontally, the pulsation k having the value $\sqrt{\frac{g}{R}}$.

The corresponding period is 84 meters. This is the period of oscillation, on a perfectly flat and polished billiard table, horizontal at its center, of a billiard ball free of friction. In the vertical direction, inertial navigation is unstable, with the coefficient occuring in the exponential having the value $\sqrt{\frac{2g}{R}}$.

3.4

The properties discussed above, with respect to accuracy, must obviously play a predominant role in any study of the applications of inertial navigation. One might pose the question, a priori, as to whether the imposibility of measuring that part of the acceleration which is due to gravity, and the need to substitute for such measurement a knowledge of this acceleration as a function of position, constitutes a strength or weakness of inertial navigation. The answer is a guarded one: everything depends on the structure of the gravitational field. In a direction in which the gravity gradient is negative, this circumstance limits the error; in one in which the gravity gradient is positive, it amplifies it. In the case of three-dimensional navigation, there are always directions for which the error will be amplified.

Inertial navigation will therefore be limited to applications of two types:

- three-dimensional applications, but rigorously limited in time in order to maintain acceptable accuracy;

- long-duration applications, but of a limited number of dimensions in order to assure stability.

4. Angular Navigation /13

4.1

The time has come to return to a problem which we have temporarily disregarded: the problem of angular navigation. We have already noted that this procedure can be reduced quite simply to a series of star sightings. Because of their enormous distance in comparison with the range of even our most ambitious space vehicles, stellar bodies constitute quasi-perfect directional references. The visibility of these stars, however, is far from being universally assured. The determination of their direction - particularly if reliance is to be placed in automatic equipment - entails a considerable number of difficulties relating to the very faint radiation energy level which serves as an indication of their presence. It would therefore appear to be of interest to resolve the problem of angular navigation itself by means of experiments interior to the system. Ideal, unrestricted endonavigation may be realized in this manner.

One might conceive of attacking the problem through a simple transposition of linear navigational methods. The study, using a suitable number of well distributed accelerometers, of the apparent gravity field on board a vehicle, or the direct employment of angular accelerometers, the operational principle of which obviously derives from that of the linear accelerometer, would permit the measurement of the angular accelerations of the vehicle and the deduction therefrom, by double integration, of the quantities which define its orientation - Euler angles, for example. But the principles of mechanics enable us to do far better.

In the expression which defines the apparent gravity we have in fact taken note of the presence of the term $- 2\vec{\omega}_e \wedge \vec{V}_r$, corresponding to the Coriolis acceleration. This means that the angular velocity $\vec{\omega}_e$ of the vehicle can be demonstrated by an experiment in mechanics carried out on board. More precisely, the difference between the sensible force required to keep at rest the mass m and that which imparts to it a velocity $\vec{V}_r$, constant and known, provides two components of $\vec{\omega}_e$. A second experiment of the same nature, involving a velocity $\vec{V}_r$ of different direction, accomplishes the determination of this vector. The accuracy of the measurement obviously increases with the relative velocity. Thus, on-board experimentation with material masses in relative rapid movement assumes a fundamental importance and leads us quite naturally to a consideration of the paramount role of the gyroscope.

4.2 /14

In fact, if one wishes to experiment with a mass m, in rapid movement, without having it almost instantaneously escape the control of the experimentor, there is practically speaking no other alternative but to impart to it an oscillatory or circular motion. The first, or oscillatory, principle, which is occasionally recommended, results in the use of vibrators similar to tuning forks. However, it is the second approach which is almost universally adopted. By experimenting with a simple revolving rotor, rapidly rotating about its axis and suspended without friction about its center of inertia, we can in fact approach this problem in an altogether satisfactory manner. The maintenance of the circular movement of the elementary masses is assured by the simple play of the solid couplings of the rotor; the effects on these elementary masses of the static term of the apparent gravity are destroyed, and those of the Coriolis term are added. The theory of the device resolves to the application of the kinetic moment theorem which, by means of the gyroscopic apprximation, is expressed by the well-known formula:

$$\vec{C} = \vec{\omega} \wedge I\vec{\Omega} \tag{6}$$

where $I\vec{\Omega}$ is the kinetic moment of the gyroscope, $\vec{\omega}$ is the angular precession velocity of its axis, $\vec{C}$ is the external coupling which is applied to it.

Let us install a gyroscope of this kind on board our vehicle, subject to such linkages that it will accompany the vehicle in its translational movement, but will be entirely free in rotation, disregarding the action of a coupling $\vec{C}$, which is controllable and measurable. Let us at every instant adjust the value $\vec{C}$ so that the axis of the kinetic moment remains aligned along a fixed direction of the vehicle. The measurement of $\vec{C}$ and the knowledge of $I\Omega$ will enable us to define, at every instant, the component of $\vec{\omega}_e$, normal to that direction. A second experiment, involving a direction different from the first (in practice, perpendicular), will conclude the determination of $\vec{\omega}_e$ itself. The device thus realized constitutes a gyrometer. In actual practice, for reasons of precision, we will prefer to use gyrometers of a single degree of freedom, which furnish only one component of the velocity $\vec{\omega}_e$. Three gyrometers of this type, with their sensitive axes oriented, respectively, along with three axes of the moving trihedral, will provide the three $\vec{\omega}_e$

components along these three directions, and we shall be able to arrive at the quantities which define the orientation of the trihedral by simple integration.

4.3 /15

There is another way of attacking the problem. Let us imagine that we place on board a vehicle a solid moving frictionlessly about its center of inertia, situated, at the moment of departure, in a known orientation and without initial angular velocity. This solid will preserve the same orientation during the entire voyage. It will obviously solve the problem of angular navigation, whether it is employed to determine the orientation of the vehicle with respect to it, or whether there are mounted on the solid itself linear accelerometers to furnish directly the acceleration components in the fixed axes. However, this procedure, although theoretically conceivable and occasionally proposed, would be unacceptable in actual practice for considerations of accuracy. It is, in fact, characterized by the proportionality of the second derivative of the angular parameters to the coupling perturbation, with all the troublesome consequences that this entails. Once acquired, an angular velocity develops a disorientation which increases with time, even if the perturbing coupling has disappeared; a constant source of perturbation in the coupling produces a deviation which increases as the square of the time. Inherent in the method are the same defects which attach to the measurement and integration of angular accelerations.

Let us replace the solid without initial velocity by a solid which this time is invested with a very great velocity of rotation about its center of revolution and is likewise suspended without friction at its inertial center. We now have a free gyroscope. A free gyroscope moves only in the direction of its kinetic moment, and in that direction alone. It will thus be necessary to supplement it with a second device, of kinetic moment not parallel to the first (in practice, perpendicular), in order to accomplish the determination of the vehicle's orientation. However, the direction thus indicated is infinitely less subject to and affected by perturbations than in the preceeding case. Reasoning, for example, on the basis of a constant perturbing coupling, it is immediately evident from formula (6) that the increase in the deviation as a function of time will be linear and not parabolic, and that the abscissa of the intersection of the straight line and the parabola is given, for equal inertia, by $\Omega t = 1$. This is the time, extremely short, which the gyroscope requires to turn one radian. Thus, it is practically certain, considering the operating speeds attainable by gyroscopes, that the ratio of the deviations in the two cases, increasing linearly with time, will already be in the order of several thousandths of a second following the application of the perturbing couplings. Another consequence of formula (6) is obviously the arresting of the deviation when the coupling perturbation is suppressed. The measurement of previous angular velocities is cancelled.*

*This cancellation is of course only apparent and no velocity discontinuity, which would violate the principles of mechanics, takes place. When the perturbing coupling is suppressed, the movement of the gyroscope axis does not disappear but winds along a cone of imperceptible angle.

The practical unfeasibility of providing an on-board platform with permanent orientation secured through inertia alone does not negate the interest which the installation of such a platform arouses. What it does do is to demonstrate that permanence of orientation must of necessity be achieved through a follow-up monitoring system. The platform will have to carry its own orientation reference instrumentation, free gyroscopes or gyrometers. Points of articulation will be motor-drive in order to combat the angular deviations - or angular velocities - detected by these instruments. Installed on the platform will be accelerometers to provide direct readings of the sensible acceleration components in the absolute axes. /16

4.4

Finally, an inertial navigation system may be designed to employ one of two extreme concepts. The first consists in linking the accelerometers to the moving trihedral, whose orientation is reconstructed either by integration of gyrometers also bound to the moving trihedral or by reference to free gyroscopes.

The second method consists in placing the accelerometers on a stabilized platform, whose permanent orientation is assured by a phase-locked monitoring system based on the readings of gyroscopes mounted on the table. As a variant approach, the stabilized platform may be subject to laws of orientation other than that of parallelism to an absolute trihedral, provided its angular evolution remains slow and controlled.

The second system is the one almost always employed. The fact is that it embodies considerable advantages. On the one hand, computations are reduced to a minimum. On the other, the angular-detection gyroscopic equipment functions in a zone of very low amplitude around zero - a circumstance eminently favorable to its precision. Conversely, this system is heavier and of greater mechanical complexity. During the planning stages for the inertial navigation equipment for the US "Apollo" lunar exploration project a comparison of the two systems was thoroughly debated. The decision ultimately reached will preserve the conventional controlled-platform system as the primary equipment, backing it up with an emergency unit operating in axes bound to the vehicle.

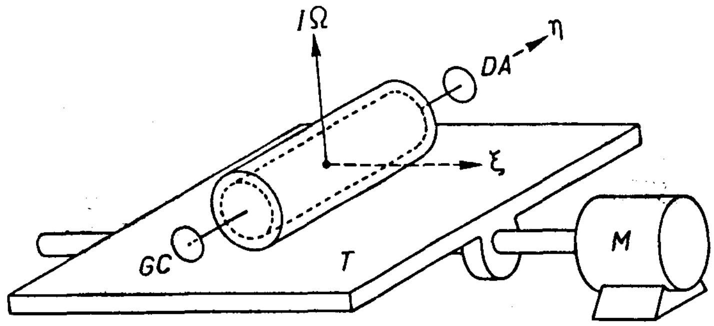
4.5.

Whatever the system type adopted, gyroscopic instrumentation plays a basic role. The design of high-precision gyroscopes is one of the indispensable conditions of inertial navigation.

In mechanical terms, the art of gyroscopy may be defined very simply: it is the problem of applying to a solid a system of forces whose geometric sum is /17
such as to ensure the "suspension" of the solid (that is, the translational comovement of the solid with and by the vehicle) and whose resultant moment with respect to the center of inertia of the solid remains zero as rigorously as possible.

The utilization of Newtonian forces, bound by nature to the mass, would be highly desirable, if it were possible. However, such forces - except for the infinitesimally small effect that may be derived from the displacement of near-by masses - are essentially uncontrollable. It is thus necessary to employ force-generation phenomena having no natural bond to mass. Coupling nullity can be achieved only by common reference to a precise geometric configuration, to which are tied in the strictest possible manner both the center of inertia and the torsional stress. The first condition poses a metrologic problem of balancing, common to all gyroscopic techniques. The second imposes the need for very careful selection of force-generating phenomena.

The utilization of material bearings - that is, the involvement of contact forces between machined solids - was the first gyroscopic suspension technique, in an effort to minimize, through the use of ball bearings as perfect as possible, the unfavorable effect of friction, which was the specific liability of this approach. Important progress was made with the appearance of the floating gyroscope, a technique foreshadowed by the gyroscopic compass of marine navigation. Suspension is ensured by the play of pressures brought to bear by a liquid on the geometric surface of reference. If the apparent gravity is uniform, (this always being the case on a stabilized platform), if the center of inertia of the suspended mass coincides exactly with the keel center, and finally if the mean density of the suspended body is equal to that of the liquid, the torque of the pressures on the keel automatically ensures the translational driving or co-movement of the floating body without the need for any coupling whatsoever. Any viscosity on the part of the liquid obviously invalidates this conclusion, but viscosity has the advantage over dry friction of cancelling itself with velocity. Thus, a floating gyroscope utilizing a liquid of as low a viscosity as possible, working about zero, may make an excellent free gyroscope. Viscosity has the additional advantage of being capable of precise determination and of producing effects which are exactly linear. A floating gyroscope using a liquid of non-zero but well-defined viscosity may function as an integrating gyrometer with no external intervention or as a gyrometer with one degree of freedom if its movement is prevented by a controlled coupling. Proper operation as an integrating gyrometer makes mandatory the use of a stabilized platform, since the integration of angular velocity about an axis cannot be exploited unless this axis has itself a quasi-fixed direction (Fig. 2).



/18

Fig. 2.--IΩ: kinetic moment; DA: angular detector; GC: coupling generator; T: table; M: motor.

The floating gyroscope is at the heart of present-day inertial navigation applications. Other suspension techniques, aimed at further improving gyroscope performance, are in the study or development stage. The majority of these techniques are based on the following idea. If we are able to set to work strictly normal surface forces and if we apply these forces to a spherical surface, we can be certain of the existence of a resultant passing through the center of the sphere, whatever may be the distribution of pressures at the different points of the surface. The normal forces may be:

- non-viscous fluid pressures in any movement, leading to the concept of the pneumatically suspended gyroscope;

- electrostatic actions on conductors. Here, the corresponding variant is the electrostatically suspended gyroscope. This type of suspension is suitable only for moderate apparent gravity and requires some sort of monitoring system (due to the absence of a natural equilibrium, which is incompatible with the properties of an electrostatic field);

- electromagnetic actions. However, the reduction of such actions to normal surface forces is automatic only by means of a conductor completely devoid of resistivity, involving recourse to superconductivity, with all the technological complications which that entails. In compensation, natural stability for the suspended body is acquired.

Whatever the suspension technique employed, the problem of the centering correction retains the same high degree of importance. A structural imbalance will lead to deviations which are proportional to the sensible acceleration. An imbalance due to deformation of the moving apparatus under load results in errors proportional to the square of the acceleration. However, this effect can be avoided if the condition of isoelasticity is fulfilled. The elastic displacement of the center of inertia then takes place in the same direction as the resultant of the torque and all coupling is consequently eliminated.

4.6. /19

Comparing the properties of angular and linear navigation, two essential differences are evident:

- on the one hand, angular velocities are measurable by internal experiments, while it is only the linear accelerations that have this capability. Thus, the necessary preliminary stage of angular navigation involves only a single additional integration, and not two, and the entire endonavigation problem is one of the third order;

- on the other hand, there is the possibility in principle, in the case of angular navigation, of avoiding all measurements and integration through the device of a reference frame parallel to the absolute reference frame which, if so desired, can be viewed as a perfect mechanical integrator of velocities or accelerations. If the same operation were to be attempted in the case of linear navigation, it would be necessary to free the test mass whose function we have stipulated as that of following the movement of the vehicle. This free mass would constitute a perfect acceleration integrator. However, this procedure is not acceptable because the mass would leave the vehicle and the determination of its coordinates would not be an internal operation. One might just as well, therefore, take direct readings of the characteristic points of the fixed trihedral. However, there is one important exception. If the trajectory which we are seeking to impose on the vehicle is one of free fall, that is an orbit of the gravitational field in which it is navigating, the sensible acceleration must be zero at every instant. In principle, guidance can be directly ensured by freeing the test mass and maneuvering the craft in such ways as to maintain this mass in a state of relative rest. In this way, guidance of a very high order of accuracy is achieved, thanks to the feasibility of a relative navigation providing direct access to the discrepancies between the effective trajectory and the desired trajectory, which can, by way of exception, be realized. A classical example of the employment of this method can be seen in the case of certain aircraft especially designed for the experimental study of weightlessness, which are maneuvered by the pilot in the manner described above, using s simple ping-pong ball as a test mass. The method has also been exploited in a US program aimed at achieving, for research purposes, satellite trajectories totally free of any non-gravitational perturbation.

4.7. /20

It might be well at this point to examine the effect which the errors of angular navigation are likely to exert on the precision of the overall navigational process.

Regardless of the specific modalities underlying the application of this form of navigation, it will be characterized by an angular velocity vector error, the effect of which on the orientation parameters will be more or less cumulative. The result will be increasing errors for the acceleration components, which will produce increasing errors in navigation, even when we have determined that such navigation is stable.

In view of the obligatory deviation introduced by inertial angular navigation, there is consequently no such thing as strict endonavigation exempt from secular errors. Naturally, however, in cases in which the navigation is stable, the increase of the errors will be far slower. A constant error in an angular velocity term is reflected in this case in a linear increase of the position error.

If the reference trihedral (materialized or not), which is supposed to remain parallel to the absolute trihedral, has turned by the small rotation vector $\vec{\xi}$, resulting from a fluctuating error $\delta\vec{\omega}$ in the angular velocity vector, we have $\vec{\varepsilon} = \int \delta\vec{\omega}\, dt$.

With $\delta\gamma_0$ the instrument error with respect to $\vec{\gamma}$, the error equation is written:

$$\frac{d^2\delta\vec{r}}{dt^2} - \overline{\text{grad}}\; g\; \delta\vec{r} = \delta\vec{\gamma}_0 - \vec{\varepsilon} \wedge \vec{\gamma}. \qquad (7)$$

It is possible to establish an interesting general property with respect to the reconstitution of the local $\vec{g}$ vector. The error for $\vec{g}$ includes an error in computation for the absolute components $\delta\vec{g}_1$ and a supplementary error committed by the navigator in transfering these components onto deviated axes. We derive directly from Eq. (3):

$$\frac{d^2\delta\vec{r}}{dt^2} = \delta\vec{\gamma} + \delta\vec{g}_t = \delta\vec{g}_1 + \delta\vec{\gamma}_0 - \vec{\varepsilon} \wedge \vec{\gamma}$$

On the other hand: $\delta\vec{g}_2 = \vec{\varepsilon} \wedge \vec{g}$

whence:

$$\delta\vec{g}_1 + \delta\vec{g}_2 = \frac{d^2\delta\vec{r}}{dt^2} - \delta\vec{\gamma}_0 + \vec{\varepsilon} \wedge \vec{g} + \vec{\gamma} \qquad (7)$$

and

$$\delta\vec{g} = \vec{\varepsilon} \wedge \frac{d^2\vec{r}}{dt^2} + \frac{d^2\delta r}{dt^2} - \delta\vec{\gamma}_0. \qquad (8)$$

The first term alone contains a secular part, on the further supposition that $\frac{d\vec{r}}{dt}$ increases indefinitely. In many cases, when $\frac{d^2\vec{r}}{dt^2}$ has a mean value of zero or is small with respect to $\vec{g}$, Eq. (8), practically speaking, ensures for $\delta\vec{g}$ a very /21 small value. The navigator then reconstructs a vector $\vec{g}$ which varies very little in space from the true gravity vector.

5. Navigation in Bound Movements

Following the definition of the term in mechanics, by bound movement we understand a case in which the moving object is compelled to remain on a curve or surface, either naturally because of a bond or artifically because of a system of navigation which does not call upon inertial procedures. The first case is illustrated by a ship; the second by an aircraft which maintains its altitude with the help of an altimeter, or a submarine.

5.1.

Let us first consider navigation on a known curve (Fig. 3).

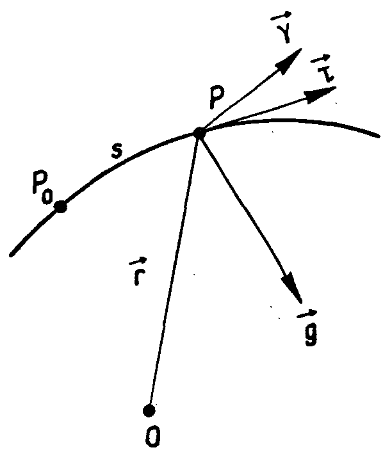


Fig. 3

The navigational computation will consist in identifying the tangential acceleration at $\frac{d^2s}{dt^2}$; whence the equation:

$$\frac{d^2s}{dt^2} = \vec{\gamma}\cdot\vec{\tau} + \vec{g} \cdot \vec{\tau}$$

where $\vec{\tau}$ is the unit vector of the tangent, a known function of s.

Let us turn to a consideration of the stability. Let $\delta\vec{\gamma}$ be the error for $\vec{\gamma}$ which causes the error δs for s. We have: /22

$$d^2 \frac{\delta s}{\delta t^2} = (\vec{\gamma} + \vec{g}) \frac{d\vec{\tau}}{ds} \delta s + \vec{\tau}\delta\vec{\gamma} + \vec{\tau}\, \overline{\overline{\text{grad}}}\, g \cdot \vec{\tau}\delta s$$

Now, $\frac{d\vec{\tau}}{ds} = \frac{\vec{n}}{R}$, R being the radius of the curve and $\vec{n}$ the unit vector of the principal normal.

On the other hand:

$$(\vec{\gamma} + \vec{g}) \cdot \vec{n} = \frac{1}{R}\left(\frac{ds}{ds}\right)^2 = \frac{V^2}{R}.$$

whence:

$$\frac{d^2}{dt^2}\delta s - \left[\vec{\tau}\, \overline{\overline{\text{grad}}}\, \vec{g} \cdot \vec{\tau} + \frac{V^2}{R^2}\right] \delta s = \vec{\tau} \cdot \delta\vec{\gamma} \qquad (9)$$

If the velocity is small, the stability criterion is the same as in rectilinear unidimensional navigation. However, the velocity exercises a destabilizing influence. The condition of stability is written:

$$\frac{V^2}{R^2} < - \vec{\tau} \cdot \overline{\overline{\text{grad}}}\ \vec{g} \cdot \vec{\tau}$$

$\vec{\tau}\ \overline{\overline{\text{grad}}}\ g \cdot \vec{\tau}$ is the gradient of $\vec{g}$ in the direction of $\vec{\tau}$. It depends only on the tangent to the curve. It must not be confused with the derivative from from s of the tangential component $\vec{g} \cdot \vec{\tau}$ of the gravity (at the surface of the Earth the first quantity equals $\frac{g}{R}$; the second - zero).

5.2.

Let us consider navigation on a surface. The general problem is quite complicated. In a simple case, when the movements are slow and hence the acceleration normal to the surface is small the study of the stability of navigation about a point can be likened to the study of navigation in the tangent plane. It is therefore the structure of the gravity gradient in the tangent plane that is decisive in terms of stability.

If the surface is a level surface of the gravity field and if it is convex like the terrestrial geoid, stability of linear navigation is everywhere ensured. If the angular navigation is inertial, the deviation of the angular reference frame comprises the stability, as we have already had occasion to note.

6. Navigation with a Non-Galilean Reference /23

6.1.

It may be desirable to navigate with a non-Galilean frame of reference - the Earth, for example, whose absolute movement is itself known. It is always possible, in this case, to begin by fixing the position of the moving body with respect to a Galilean mark with a subsequent change of reference by means of a simple transformation of coordinates. However, this method may be detrimental to the accuracy of the calculations, particularly in a case in which the relative displacement of interest to the navigator is small in amplitude with respect to the absolute and driving displacements, of which it constitutes the difference.

It may be preferable in this case (and there is no impediment) to base the navigational computation on the driven reference, which we shall refer to as the navigation reference. Quite clearly, all that is required is to substitute, in Eq. (4), the Newtonian attraction $\vec{g}$ for the apparent gravity in this system. The equation to be solved will then be the following:

$$\frac{d^2\vec{r}}{dt^2} - \vec{g}_a\ (\vec{r}) + 2\vec{\omega}_e \wedge \frac{dr}{dt} = \vec{\gamma}\ (t). \qquad (10)$$

The static apparent gravity $\vec{g}_a = \vec{g} - \vec{g}_e$ is a well-defined function of $\vec{r}$ (possibly of time also); $\vec{\omega}_e$, the driving rotation, may also be a function of time. However, if the movement of the navigation reference is a constant rotation about a point in uniform movement (the case of the Earth with good approximation), time does not figure in the first member.

There are several precautions to be taken with regard to angular navigation. If gyroscopes are employed which are bound to the axes of the vehicle the relative rotation velocity of the instrument must be calculated by the difference between the absolute measured rotation velocity and the drive rotation velocity which is known a priori, with subsequent integration in order to determine the orientation of the vehicle with respect to the navigation reference.

If a platform is used, it is essential to maintain it parallel to the navigation reference, in order to measure directly the accelerations by their components in the calculation axes. There will be imparted to the platform the drive rotation velocity $\vec{\omega}_e$ thanks to the coupling motors with which the gyroscopes are equipped. The platform can be readjusted with respect to stars of known directions at every instant in the navigation reference.

It is of interest to examine the case of gravimetric navigation with a non-Galilean reference. The relative rest characterized by $\frac{d\vec{r}}{dt} = 0$ gives:

$$- \vec{g}_a = \vec{\gamma}.$$

The measurement of $\vec{\gamma}$ thus defines a point of the reference with which the vehicle coincides. It is still necessary, of course, to know the orientation with respect to the vehicle of the navigation reference, or to have a platform which is parallel to the axes of this reference. /24

It might be noted, however, that even should reference to the navigation mark be lost, useful information can still be obtained if one assumes that it is possible to achieve relative angular rest of the vehicle (or of the platform) with respect to the navigation reference frame. The absolute angular velocity of the vehicle, which can always be measured, is then identified with the rotation velocity $\vec{\omega}_e$ of the navigation reference. It is thus possible to measure in the axes of the vehicle both $\vec{\omega}_e$ and $\vec{g}_a$. One is thus able to find one's position at points in the navigation reference characterized by a given value of the angle formed by these two directions.

These principles, applied to navigation on the surface of the globe, lead to methods which have been known and in use for a long time:

- determination of latitude and longitude by fixing the local vertical at a preciselyknown hour;

- the possibility, by a purely inertial experiment, of determining the direction of the pole with respect to the axes of a vehicle at rest, or of the local axes bound to the reference (this being the purpose of the gyroscopic compass);

- determination of latitude by an inertial experiment, latitude being by definition the compliment of the angle formed bu the apparent gravity and the line of the poles, that is the vector $\vec{\omega}_e$.

The gyroscopic compass has long been in use at sea. On the contrary, the determination of latitude by means of the properties of the gyroscope is not widespread. It is easy to see that latitude determination is in fact more difficult than the determination of course, so much so, indeed, that for a long time the technique has been held to be utopian. If we concede that the essential perturbations in a gyroscope are the result of centering defects, it will be recognized that the perturbing couplings have axes which are essentially horizontal. In gyrometry experimentation, therefore, it is always of advantage to employ vertical control couplings. While this is possible in the measurement of a horizontal angular velocity, it is impossible in the measurement of a vertical angular velocity, for the reason that the control coupling and the component of the measured angular velocity are necessarily orthogonal. The horizontal components of $\vec{\omega}_e$, whose ratio determines the course heading, are therefore better known than the vertical component, which is required for latitude determination.

7. Inertial Navigation Applications /25

Ballistic Missile Applications

Having examined the fundamental principles of inertial navigation, let us briefly review its different areas of application, beginning with the case of the ballistic missile.

We have seen that three-dimensional inertial navigation is always subject to instability and must therefore be limited to very brief intervals of time. The passage through the atmosphere offers, in particular, an opportunity favorable to the use of this method. The optimal stiuation called for by economy of propulsion as well as thermal and mechanical structural resistance leads to accelerations in the order of 100 m/sec^2 acting during a period of time in the order of one minute. Thereafter, the missile is located in a vacuum and is subject only to the force of gravity. Inertial navigation restricted merely to the verification that $\vec{\gamma} = 0$ is evidently of little use during this period. Its role consists in furnishing, at the termination of the powered-flight stage, the initial conditions which will permit the calculation of the subsequent Keplerian trajectory. Among these initial conditions, velocity is of greater importance than position, this fact being an aid to accuracy, since velocity is obtained through a simple integration.

Inasmuch as the operating time is small with respect to the 84-minute period which characterizes stability in the horizontal sense, as well as with respect to the time constant, of the same order of magnitude, which describes the instability in the vertical sense, neither this stability nor this instability play any practical role, the errors being those which result from the integration of $\delta\vec{\gamma}$.

The inertial installation platform can be kept parallel to the absolute axes, if it is preferred to carry out the computations in these axes. The platform may also follow, in the diurnal movement of the Earth-bound axes with which it coincided at the moment of launching. The initial precision adjustment of the platform obviously poses a difficult problem, all the more if the missile is fired from a surface vessel or submarine.

An eventual resumption of inertial navigation during re-entry into the atmosphere has been occasionally discussed. The principal difficulty in this connection derives from the attitude of the platform, which continues to deviate during the entire Keplerian flight. Stellar recalibration before atmosphere would be highly desirable.

8. Horizontal Navigation Applications

8.1.

This application pertains to the navigation of a ship, aircraft, or submarine which must follow, strictly or approximately, the terrestrial geoid, with the distance to the geoid, if it is not automatically zero, measured non-inertially.

The problem thus concerns a case of surface-bound navigation. To simplify matters, let us limit ourselves to the study of navigation on a great circle arc, imagining the Earth to be spherical and, for the time being, motionless (Fig. 4). /26

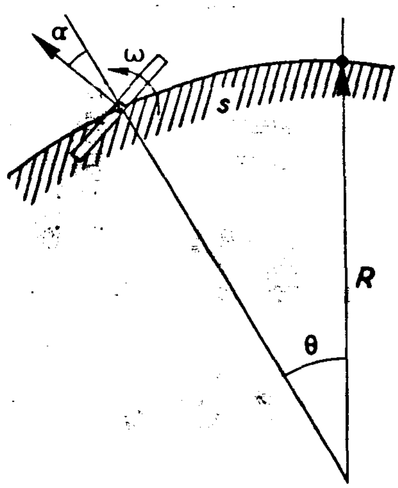


Fig. 4

Since the tangential component of $\vec{g}$ is zero, the navigational computation is reduced to $\frac{d^2s}{dt^2} = \vec{\gamma} \cdot \vec{\tau}$ ($\vec{\gamma} \cdot \vec{\tau}$ is the sensible horizontal acceleration). To obtain $\vec{\gamma} \cdot \vec{\tau}$, we could of course measure all the components of $\vec{\gamma}$ in an arbitrary system, a stabilized platform for example, and project them on the tangent to the circle. It is obviously simpler to eliminate the measurement of the vertical component of $\vec{\gamma}$, since it need not concern us at all. This brings us to the use of only two accelerometers (one in the plane problem) carried by a platform which must obviously be maintained rigorously parallel to the local horizontal plane. If this condition is fulfilled at departure, it will be preserved, provided there is imparted to the platform an angular velocity such that:

$$\omega = \frac{d}{dt} = \frac{1}{R}\frac{ds}{dt} = \frac{1}{R}\int \gamma \, dt \qquad (11)$$

setting $\gamma = \vec{\gamma} \cdot \vec{\tau}$, and being the meridian arc corresponding to arc s.

This operation can be carried out by a one-time integration of the acceleration measured on the platform, by calculating ω by the formula given above, and by imparting to a gyrometer mounted on the platform the precession velocity ω, thanks to the use of an appropriate precession coupling. Since the platform is phase-locked to the moving gyrometer housing, the angular velocity ω is certain to be imparted to the platform itself.

This procedure thus provides a means of determining the local vertical. /27
The navigation problem can now be resolved:

- either by determining this vertical with respect to the stars, if they are observable;

- or by installing on board the vehicle a supplementary platform, stabilized in absolute space, perpetuating the initial orientation of the forst platform;

- or by integrating the rotation velocity ω measured by the gyrometer, or by re-integrating the acceleration γ measured by the accelerometer.

The last two methods, which respect the principle of endonavigation, are fundamentally equivalent; both introduce an error which increases with time.

8.2.

Let us now consider the accuracy of the operation. The system will not be perfect unless the platform was horizontal and without angular velocity at launch, and unless the measurements were rigorous. In point of fact, the table will make a small angle α with the vertical (Fig. 4). The accelerometer, instead of strictly measuring the horizontal acceleration of the moving vehicle, is also sensitive to the vertical reaction component, following the platform, which ensures that the movement will be circular. The measured acceleration γ^* thus has the value:

$$\gamma^* = \gamma - \alpha \left[g - \mathrm{R}\left(\frac{d\theta}{dt}\right)^2\right] + \delta\gamma$$

and the measured angular velocity ω^* is written:

$$\omega^* = \omega + \delta\omega.$$

In these formulas, $\delta\gamma$ and $\delta\omega$ represent the instrument errors due to the accelerometer and the gyrometer.

Relation (9) evidently applies to the measured parameters, whence:

$$\int \left\{ \gamma - \alpha \left(g - \mathrm{R}\left(\frac{d\theta}{dt}\right)^2\right) + \delta\gamma \right\} dt = \mathrm{R}\,(\omega + \delta\omega).$$

Integrating once and bearing in mind that:

we obtain
$$\gamma = R\frac{d^2\theta}{dt^2} \qquad \omega = \frac{d}{dt}(\theta + \alpha)$$

$$\frac{d^2\alpha}{dt^2} + \alpha\left[\frac{g}{R} - \left(\frac{d\theta}{dt}\right)^2\right] = \frac{\delta\gamma}{R} - \frac{d\,\delta\omega}{dt}. \tag{12}$$

This formula permits the calculation of platform error * . In the absence of instrument errors in the measurement of γ and ω, it will be seen that the platform oscillates with a periodicity defined by the pulsation

$\sqrt{\frac{g}{R} - (\frac{d\Theta}{dt})^2}$.

For the low speeds of ships, $(\frac{d\Theta}{dt})^2$ is practically negligible with respect to $\frac{g}{R}$. The period is then approximately 84 minutes. /28

It can be very easily demonstrated that a pendulum designed to have an 84-minute period at rest (Schuler pendulum) provides a perfect replacement for the controlled platform. When its point of articulation is shifted, it functions as an exact mechanical integrator of the inertial navigation equation. In actual practice, such a pendulum is not feasible, since it would require a suspension not possible at the present-day state of the art.

The role of the instrument errors is to excite the platform oscillations at the Sculer frequency - oscillations which exist even when such errors are absent, unless the initial conditions $\alpha = 0$, $\frac{d\alpha}{dt} = 0$ are strictly met.

Let us now consider the navigation error $\delta\Theta$. The value of Θ provided by the instrument, or Θ^*, may be calculated by integrating ω^* or $\int \frac{\gamma^*}{R}dt$. The equality of these two quantities is ensured by definition, and thus the values found for Θ^* will evidently be identical. Take, for example:

$$\theta^* = \int \omega^* dt = \theta + \int \delta\omega\, dt + \alpha.$$

This gives:

$$\alpha = \delta\theta - \int \delta\omega\, dt.$$

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*The fact that α is devoid of secular error is a consequence of Eq. (8), which indicates that the error in vector $\vec{g}$ does not affect its direction. This property is not strictly generalizable to bidimensional navigation, since the latter introduces a vertical component of $\vec{\varepsilon}$.

whence, by substituting in Eq. (9):

$$\frac{d^2\delta\theta}{dt^2} + \delta\theta \cdot \left[\frac{g}{R} - \left(\frac{d\theta}{dt}\right)^2\right] = \frac{\delta\gamma}{R} + \left[\frac{g}{R} - \left(\frac{d\theta}{dt}\right)^2\right] \cdot \int \delta\omega d\theta. \quad (13)$$

This equation does not differ from that provided by the general theory of navigation on a curve. The second member presents, in fact, the tangential component of the total error committed in the measurement of the acceleration. Had we measured by comparing the orientation of the moving platform with that of a platform materializing the absolute axes, we might also have written:

$$\delta\theta = \alpha + \int \delta\omega\, dt$$

$\delta\omega$ being the error of the gyrometer used to control the permanence of the orientation of the second platform. Eq. (11) remains correct, the angle /29

$$\delta\beta = -\int \delta\omega\, dt$$

being now interpreted as the angle caused by the deviation of the second platform.

8.3.

This simplified study has cast some light on the principal properties of horizontal inertial navigation. These properties may be summarized as follows:

1. Stability is ensured in the sense that there is available a platform for the determination, without secular error, of the vertical. The second number of Eq. (10) contains only quantities of mean zero value.

2. If it is impossible to fix the vertical thus marked with respect to the stars, a secular error for position is introduced through accumulation of the errors $\delta\omega$.

3. The accuracy of the operation is exactly the same, whether one proceeds by an integration of ω or by a new integration of γ. In both cases, the accuracy is affected by the errors in the two parameters. An attempt might be made to judge the two methods by introducing the platform control errors, thus invalidating the exactitude of Eq. (11). It is found that the introduction of a control time constant, for example, does not permit a universal choice between the two procedures, since their comparison depends on the law of movement.

Let us emphasize this somewhat controversial point. Horizontal inertial navigation can be presented under two forms which appear quite different. For some investigators, the position is determined by double integration of the accelerations with the proper movement imparted to the accelerometer support table in order to ensure that these accelerometers operate in the proper axes. For others, a perfect vertical is first achieved by insensitizing a pendulum to accelerations, with the position then calculated on the basis of the angle formed by this vertical as it turns *. In fact, the difference between these two presentations (each of which is correct and each of which has its own didactic value) is a purely semantic one. In any event, one is led to the same equations and to the same instrumentation, even if there is a difference in the execution of the final integration which, as we have just seen, is without influence on the accuracy of the result.

4. Stability of navigation is ensured for low velocities. It decreases at higher velocities and is finally altogether cancelled at satellization velocity. If one day a vehicle with a super-orbital velocity capability were designed to remain in the Earth's atmosphere, because of inverse lift, inertial navigation would be unstable for that vehicle. /30

Stability does not depend on the availability of a platform fulfilling the Schuler condition. It will be ensured even in the event of horizontal navigation with the aid of a platform which is stable with respect to absolute space. Stability depends only on the structure of the gravitational field.

8.4.

Let us finally introduce the rotation of the Earth. Since navigation must provide geographic coordinates, it is most simple to reason on the basis of a reference frame carried forward by the diurnal movement. The modifications brought by the rotation are the following:

The rotation imparted to the platform must be the vectoral sum of the relative rotation, calculated by integration of the relative acceleration, and the driving rotation, which is known. The relative acceleration is itself derived from the measured acceleration, which is an absolute acceleration, by subtraction of the horizontal component of the Coriolis acceleration due to the rotation of the Earth. Thereafter, relative angular velocity or relative acceleration is integrated. In practice, in order to simplify the calculations, the horizontal axes of the platform are directed toward the cardinal points.

A well-known consequence of the Earth's rotation is the disappearance of the secular error of course and latitude, the sole remaining error of non-zero mean value being the longitude error.

This seems a paradoxical result. It is not immediately clear how the adoption of a different reference of movement and system of computation can transform the accuracy of the operation. Actually, the difficulty rests in the experientially derived notion of the random structure of a measurement error. It is customarily held that the error in the measurement of a time function is the sum of a constant systematic error and of a random error having a more or less known spectral distribution. All told, what this means is that there is ascribed to the error a continuous spectrum, except for the zero frequency constituted by a line. When the magnitude measured is a velocity of rotation about

* For a very clear exposition of this second point of view and a most useful discussion of the orders of magnitude of the errors see [3].

a material axis, the systematic error is a vector carried by that axis. The resultant position error is the vectoral integral of the elementary error. If the axis is fixed, these errors are cumulative. If the axis itself turns in absolute space at an angular velocity ω about one of its normals, the position error has a mean value of zero. The cumulative effect then derives from the frequency component $\frac{\omega}{2\pi}$ of the random error, but this does not correspond to a /31 spectral line and the effect is extremely weak. An inertia platform carried by a ship turns perceptibly with the Earth, since the speed of the ship is small with respect to that of the Earth. The polar line is therefore a direction common to absolute space and to the platform. The vector $\delta\vec{\omega}$ which affects the measurement of the angular velocity of the table has a fixed projection on the line of the poles and perceptibly sinusoidal projections on the free space directions normal to the line of the poles. Thus, cumulative orientation error exists only in longitude.

This result, which is valid only in the case of a vehicle whose velocity is small with respect to that of the Earth, obviously bears a direct relation to the possibility, already mentioned, of determining the polar line by an inertial experiment in gravimetric navigation.

9. Space Applications of Inertial Navigation

We have seen that three-dimensional inertial navigation suffers from irremediable instability. It is therefore of necessity limited to periods which are short with respect to the characteristic time constants of this instability.

The space trajectories in use at the present time are indeed comprised of brief periods of powered flight separated by extensive periods of free flight. The propulsion or powered-flight intervals lend themselves to the application of inertial navigation under the same conditions as the initial launch period. It is extremely useful in ensuring that the propulsion stage will be marked by such strict adherence to prescribed parameters as will hold the need for further corrections to a minimum. The coasting periods will naturally be exploited for an adjustment of the results based on electromagnetic or optical tracking procedures. In this way, a new powered-flight segment may be initiated with updated data with respect to position and initial velocity, and with a platform oriented on the stars.

These operations, which can be performed at leisure, may be entrusted to a human navigator who will employ instruments deceptively similar to sextants and periscopes and whose function will be reminiscent of the most traditional duties of the naval officer. The role of optics in this area will be a predominant one, since, with the exception of those regions of the globe which are covered by clouds, visibility is guaranteed; even the tropical zones furnish a wealth of geographic details capable of serving as landmarks. These markers are always available since beyond a certain altitude the space navigator never loses sight of land, taking the word "land" here in its traditional maritime sense. The stars will retain their role as directional reference markers, with the closer of them gradually replacing terrestrial landmarks as the spacecraft /32 pulls away from our globe, whose ultimate contribution to position determination will be largely limited to the direction of its center and apparent diameter.

The navigational system of the already far-advanced American "Apollo" moon project is based on just such principles. As one gains familiarity with this program, one cannot help being struck by the high degree of autonomy left to the flight navigator. Although supported under normal conditions by important supplementary data from the ground, the navigator has been provided with every facility to ensure his ability to perform his mission with maximum efficiency even without such support.

10. Conclusion

In the matter of using measurement to reconstitute the variation law of a quantity as a function of time, the available instrumentation is frequently marked by certain complementary features whose very diversity brings to mind comparative associations.

Among these comparisons, perhaps the most frequent is that of an instrument capable of faithfully reproducing the rapid variations of the parameter in question but subject to a progressive cumulative deviation, with another instrument whose instantaneous error may be significant and fluctuating but whose mean accuracy shows no degeneration in time. This dichotomy is reflected, for example, in the simultaneous presence on an aircraft instrument panel of a course holding unit (auto-pilot) and a magnetic compass.

In terms of the navigational problem, the integrating accelerometer is clearly an instrument of the first type. While it is a remarkable and promising effort, there is something slightly unnatural about inertial navigation, basing, as it does, the entire position determination procedure on this one type of instrument alone. As such, the inertial technique appears justified only when direct localization is impossible - by virtue of military considerations, for example - or when it can be conveniently employed only at discrete intervals, of greater or lesser separation, during which the inertial navigational method can provide a valuable interpolation.

The remarkable success of inertial techniques in low-level circumterrestrial navigation, with its intimate relation to profound mechanical considerations, will continue to guarantee these methods an assured development for military applications. Civilian usages, on the other hand, may be less certain, in view of the tracking facilities made available by an increasingly more accurate and extensive radio navigation aid network.

In the space environment, the inherent instability of three-dimensional inertial navigation, together with the anticipated advance in optical and radio tracking systems, will necessarily result in the combined utilization of different facilities and methodologies, with the predominant function assigned to the accelerometer and the gyroscope.

Perhaps one day an untapped area will be found for the application of integral inertial navigation in the exploration of the very entrails of our planet...if man decides, after the Voyage to the Moon, to turn his attention to the Voyage to the Center of the Earth. /33

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